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Testing the α'^3 term in the non-abelian open superstring effective action

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Abstract: Recently, a proposal for the full non-abelian open superstring effective action through $\mathcal{O}(\alpha'^3)$ has been formulated in hep-th/0108169. We test this result by calculating the spectrum in the presence of constant magnetic background fields and by comparing the result to string theoretic predictions. The agreement is perfect. Other proposals for the superstring effective action through this order do not reproduce the spectrum correctly.

Keywords: Superstrings and Heterotic Strings, D-branes, Supersymmetric Effective Theories.

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Contents

1. Introduction

The tree-level effective action for a single Dp-brane is known through all orders in α', albeit in the limit of slowly varying fields. It is the $d = 10$ supersymmetric abelian Born-Infeld action, dimensionally reduced to $p+1$ dimensions [1, 2, 3]. For n coinciding Dp-branes, no all order result is known. In leading order it is the $d = 10$ U(n) super Yang-Mills action, dimensionally reduced to $p + 1$ dimensions [4]. A direct calculation requires matching the effective action to n-point open superstring amplitudes. This has been done for $n \leq 4$, yielding the full effective action through order α'^2 [5, 6, 7]. In the remainder of this paper, we will only focus on the bosonic terms in the action. In addition, we will ignore the transversal coordinates of the D-brane. The latter can be reconstructed using T-duality (see e.g. [8]).

In [8], a detailed study of the structure of the non-abelian effective action was initiated. An immediate consequence was that, as the effective action has to match gluon disk amplitudes, there is necessarily only one group trace. Furthermore, it was pointed out that the effective action $\mathcal{S}$ is given by $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3$. In this, $\mathcal{S}_1$ does not contain any covariant derivatives acting on the fieldstrength and is, by definition, the non-abelian Born-Infeld action. Both $\mathcal{S}_2$ and $\mathcal{S}_3$ contain the terms with derivatives acting on the fieldstrength, but while $\mathcal{S}_2$ has only terms with symmetrized products of covariant derivatives, $\mathcal{S}_3$ has anti-symmetrized products of covariant derivatives as well. It is clear that because of the $[D, D]\cdot = [F, \cdot]$ identity, the split between $\mathcal{S}_1$ and $\mathcal{S}_3$ is not unambiguously defined. This ambiguity was fixed in [8] by the proposal that $\mathcal{S}_1$ is the non-abelian Born-Infeld action defined by means of the symmetrized trace prescription. I.e. it assumes the same form as the abelian Born-Infeld action. Upon expanding the action in powers of the fieldstrength, one first symmetrizes all terms and subsequently one performs the group trace. Indeed, all other terms without derivatives not belonging to this class can be re expressed as elements of $\mathcal{S}_3$. In the abelian limit, $\mathcal{S}_1$ reduces then to the abelian Born-Infeld action, $\mathcal{S}_3$ vanishes and $\mathcal{S}_2$, which is present [9], vanishes in the limit of slowly varying fields. This proposal was consistent with results through order α'^2 where, modulo field redefinitions, $\mathcal{S}_2$ and $\mathcal{S}_3$ vanish.

In [10] and [11], this proposal was tested. Switching on constant magnetic background fields corresponds, upon T-dualizing, to D-branes at angles. String theory easily allows for the calculation of the spectrum of strings stretching between different branes. In the context of the effective action, the spectrum should be reproduced by the mass spectrum of the off-diagonal gauge field fluctuations. In the analysis, the spectrum was calculated using $\mathcal{S}_1$ only. Though the spectrum was correctly reproduced through $\mathcal{O}(\alpha'^2)$, it failed at higher orders. This clearly shows the relevance of the $\mathcal{S}_2$ and $\mathcal{S}_3$ terms and higher order calculations are called for. Indeed, as will become clear in this paper, such terms do contribute to the spectrum. So contrary to the abelian case, it is hard to devise a test in the non-abelian context where derivative terms can be ignored.

Recently, the next order of the effective action was obtained in [12], where an indirect approach, developed in [13], was used. A stable holomorphic bundle defines a solution of Yang-Mills theory [14], [15]. In D-brane context this corresponds to BPS configurations of D-branes in the limit of weak fieldstrengths. Requiring that such solutions continue to exist for arbitrary values of the fieldstrength determines the deformation of the Yang-Mills action. Through $\mathcal{O}(\alpha'^2)$ this yields a unique[1] result agreeing with direct calculations. At $\mathcal{O}(\alpha'^3)$ a one parameter family of solutions was obtained. The parameter could be fixed by comparing to a direct calculation of the derivative terms starting from a four point open superstring amplitude [16]. Again this result was consistent with the proposal of [8]. Indeed[2] the $\mathcal{O}(\alpha'^3)$ correction falls completely in $\mathcal{S}_2$ and $\mathcal{S}_3$.

Because of the appearance of genuine derivative terms in the effective action, one can wonder whether the spectrum in the presence of constant magnetic background fields gets correctly reproduced. In the present paper we will perform this check through $\mathcal{O}(\alpha'^3)$ and show that the action of [12] correctly reproduces the spectrum. An older direct calculation of the effective action at this order, [17], fails to do so. Recently, based on completely different grounds, another proposal was made for the effective action through this order [18]. Here as well, the spectrum is not correctly reproduced. We will comment on this on the last section.

[1] From now on, all statements we make hold modulo field redefinitions.

[2] The terms without derivatives at this order which were given in [12] can all be rewritten as terms with derivatives using the $[D, D]\cdot = [F, \cdot]$ identity.

2. The Yang-Mills spectrum

In this section we briefly recall the spectrum of Yang-Mills on tori. We start with the lagrangian,[3]

$$\mathcal{L}_{(0)} = \frac{1}{4}\,\mathrm{tr}\, F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_1}\,. \tag{2.1}$$

We compactify $2m$ dimensions on a torus and introduce complex coordinates, $z^\alpha = (x^{2\alpha-1}+ix^{2\alpha})/\sqrt{2}$, $z^{\bar\alpha} = (z^\alpha)^*$, $\alpha \in \{1,\dots,m\}$ in the compact directions. We restrict ourselves to a U(2) gauge group and switch on constant magnetic background fields in the compact directions $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\bar\alpha\bar\beta} = 0$, $\mathcal{F}_{\alpha\bar\beta} = 0$ for $\alpha \neq \beta$. So only $\mathcal{F}_{\alpha\bar\alpha}$ does not vanish and we take them in the Cartan subalgebra of $su(2)$,

$$\mathcal{F}_{\alpha\bar\alpha} = i\begin{pmatrix} f_\alpha & 0 \\ 0 & -f_\alpha \end{pmatrix}, \tag{2.2}$$

where the f_α, $\alpha \in \{1,\dots,m\}$ are imaginary constants. We write the gauge potential as, $A_\alpha = \mathcal{A}_\alpha + \delta A_\alpha$, where $\mathcal{A}_\alpha$ denotes the background and δA_α the fluctuation around it. We are only interested in the off-diagonal fluctuations as the diagonal ones probe the abelian part of the effective action. So we use the following notation,

$$\delta A = i\begin{pmatrix} 0 & \delta A \\ \delta\bar A & 0 \end{pmatrix}, \qquad \mathcal{A} = i\begin{pmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A} \end{pmatrix}. \tag{2.3}$$

In this paper, we will study the fluctuations in the compact directions. In other words only δA_γ and $\delta A_{\bar\gamma}$ are non-vanishing. Expanding eq. (2.1) through second order in the fluctuations, we find modulo a constant term,

$$\begin{aligned} \mathcal{L}^{\mathrm{quad}} &= -\delta\bar A_{\bar\alpha}\left(\mathcal{D}^2 + 4if_\alpha - \sum_\beta \mathcal{D}_\alpha(\mathcal{D}_\beta A_{\bar\beta} + \mathcal{D}_{\bar\beta}A_\beta)\right)\delta A_\alpha - \\ &\quad -\delta\bar A_\alpha\left(\mathcal{D}^2 - 4if_\alpha - \sum_\beta \mathcal{D}_\alpha(\mathcal{D}_\beta A_{\bar\beta} + \mathcal{D}_{\bar\beta}A_\beta)\right)\delta A_{\bar\alpha}\,, \end{aligned} \tag{2.4}$$

where $\mathcal{D}^2 = \mathcal{D}_\mu\mathcal{D}^\mu$ and

$$\mathcal{D}_\mu\delta A_\nu = (\partial_\mu + 2i\mathcal{A}_\mu)\delta A_\nu, \qquad \mathcal{D}_\mu\delta\bar A_\nu = (\partial_\mu - 2i\mathcal{A}_\mu)\delta\bar A_\nu\,. \tag{2.5}$$

By a gauge choice, we can put $\sum_\beta(\mathcal{D}_\beta A_{\bar\beta} + \mathcal{D}_{\bar\beta}A_\beta) = 0$. We rewrite $\mathcal{D}^2\delta A_\alpha$ in eq. (2.4), using $[\mathcal{D}_{\bar\alpha},\mathcal{D}_\beta] = -2i\delta_{\alpha,\beta}f_\alpha$ as,

$$\mathcal{D}^2\delta A_\alpha = \left(\Box_{NC} + 2\sum_\beta \mathcal{D}_\beta\mathcal{D}_{\bar\beta} - 2i\sum_\beta f_\beta\right)\delta A_\alpha\,, \tag{2.6}$$

[3] Our metric is "mostly +". Further conventions can be found in [12]. We use the indices α, β, ... for the compact (and complexified) coordinates. The indices μ, ν, ... run over both compact and non-compact directions. In addition the lagrangian, eq. (2.1), should still be multiplied by an arbitrary coupling constant $-1/g^2$. The sign arises because we use an anti-hermitian basis for the $u(n)$ Lie algebra.

where $\square_{NC}$ is the d'alambertian in the non-compact directions. So we get from eqs. (2.4) and (2.6),

$$0 = \left(\square_{NC} + 2\sum_\beta (\mathcal{D}_\beta \mathcal{D}_{\bar\beta} - i f_\beta) + 4 i f_\alpha\right) \delta A_\alpha^{\{n\}}(x)\,,$$
$$0 = \left(\square_{NC} + 2\sum_\beta (\mathcal{D}_\beta \mathcal{D}_{\bar\beta} - i f_\beta) - 4 i f_\alpha\right) \delta A_{\bar\alpha}^{\{n\}}(x)\,. \tag{2.7}$$

It is clear that in order to obtain the spectrum, we need to diagonalize $\sum_\beta \mathcal{D}_\beta \mathcal{D}_{\bar\beta}$. This was performed in [19] and [20]. A complete set of eigenfunctions, $f(z, \bar z; \{n\})$, is given by

$$f(z, \bar z; \{n_1, n_2, \dots, n_m\}) = \mathcal{D}_{z^1}^{n_1} \mathcal{D}_{z^2}^{n_2} \cdots \mathcal{D}_{z^m}^{n_m} g(z, \bar z)\,, \tag{2.8}$$

with $n_1, n_2, \dots, n_m \in \mathbb{N}$ and

$$g(z, \bar z) = e^{-2i \sum_\gamma z^{\bar\gamma} \mathcal{A}_{\bar\gamma}(z)} h(z)\,. \tag{2.9}$$

The holomorphic function $h(z)$ should obey certain periodicity conditions and was explicitly constructed in terms of θ-functions in [20] and [19]. As $\mathcal{D}_{\bar\gamma} g(z, \bar z) = 0$, $\forall \gamma$, one easily obtains the eigenvalues. We write

$$\delta A_\mu(x, z, \bar z) = \sum_{\{n\} \in \mathbb{N}^m} \delta A_\mu^{\{n\}}(x) f(z, \bar z; \{n\})\,, \tag{2.10}$$

where x denotes the non-compact coordinates and $\{n\} = \{n_1, n_2, \dots, n_m\}$. From this we immediately get the dispersion relation and as a consequence the spectrum,

$$M^2 = 2i \sum_\beta (2n_\beta + 1) f_\beta \pm 4 i f_\alpha\,. \tag{2.11}$$

String theory yields a spectrum of exactly the same form but the field strength gets replaced everywhere by [21] (see also [11]),

$$f_\gamma \to \frac{1}{2\pi\alpha'} \operatorname{arctanh}(2\pi\alpha' f_\gamma)\,. \tag{2.12}$$

So for $2\pi\alpha' f_\gamma$ small, we get agreement. The higher order corrections to the effective action should be such that the Yang-Mills spectrum gets deformed as in eq. (2.12).

3. The spectrum through $\mathcal{O}(\alpha'^2)$

At order α' only one term appears,[4]

$$\mathcal{L}_{(1)} = 2\pi\alpha' \xi \operatorname{tr}\left(\left(D_{\mu_3} D^{\mu_1} F_{\mu_1}{}^{\mu_2}\right) F_{\mu_2}{}^{\mu_3}\right), \tag{3.1}$$

[4] In order to simplify the notation, we will put after this $2\pi\alpha' = 1$

with $\xi \in \mathbb{R}$. This term can be removed by a field redefinition,

$$A_\mu \longrightarrow A_\mu - \xi D^\nu F_{\nu\mu} \,. \tag{3.2}$$

However, keeping this term and performing the calculation as above, one finds the same result for the spectrum, i.e. one recovers eq. (2.7), provided one redefines the fluctuations as

$$\delta A_\mu \to \delta A_\mu - 2\xi\Big(\mathcal{D}^2 \delta A_\mu + 2[\mathcal{F}_{\mu\nu}, \delta A^\nu]\Big) \,. \tag{3.3}$$

Such redefinitions are unambiguously determined as the dispersion relation should be of the form $(\Box_{NC} - M^2)\delta A = 0$. Essential in this is the form of the derivatives with respect to the non-compact coordinates. From now on, we will completely ignore terms which are removable through a field redefinition.

We now turn to the α'^2 contribution to the effective action. It reads as,[5]

$$\begin{aligned}\mathcal{L}_{(2)} = -\,\mathrm{tr}\bigg(&\frac{1}{24} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_3}{}^{\mu_4} F_{\mu_4}{}^{\mu_1} + \frac{1}{12} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_4}{}^{\mu_1} F_{\mu_3}{}^{\mu_4} - \\ &- \frac{1}{48} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_1} F_{\mu_3}{}^{\mu_4} F_{\mu_4}{}^{\mu_3} - \frac{1}{96} F_{\mu_1}{}^{\mu_2} F^{\mu_4}_{\mu_3} F_{\mu_2}{}^{\mu_1} F_{\mu_4}{}^{\mu_3}\bigg) \,.\end{aligned} \tag{3.4}$$

Proceeding in exactly the same way as in the previous section, i.e. taking the part in $\mathcal{L} = \mathcal{L}_{(0)} + \mathcal{L}_{(2)}$ quadratic in the derivatives and varying it with respect to $\delta A_{\bar\alpha}$, we obtain,[6]

$$\begin{aligned} 0 = &\bigg(\Box_{NC} + 2\sum_\beta \Big(1 + \frac{1}{3} f_\beta^2\Big) \mathcal{D}_\beta \mathcal{D}_{\bar\beta} - 2i \sum_\beta \Big(f_\beta + \frac{1}{3} f_\beta^3\Big) + 4i\Big(f_\alpha + \frac{1}{3} f_\alpha^3\Big)\bigg)\delta \hat{A}_\alpha + \\ &+ \sum_\beta \Big(1 + \frac{1}{3} f_\beta^2\Big) \mathcal{D}_\alpha (\mathcal{D}_\beta A_{\bar\beta} + \mathcal{D}_{\bar\beta} A_\beta) + \mathcal{O}(\alpha'^4) \,, \end{aligned} \tag{3.5}$$

where

$$\delta \hat{A}_\alpha = \bigg(1 + \frac{1}{3} f_\alpha^2 - \frac{1}{6} \sum_\beta f_\beta^2\bigg) \delta A_\alpha \,. \tag{3.6}$$

The last line in eq. (3.5) can be eliminated by making an appropriate gauge choice,

$$\sum_\beta \Big(1 + \frac{1}{3} f_\beta^2\Big) (\mathcal{D}_\beta A_{\bar\beta} + \mathcal{D}_{\bar\beta} A_\beta) = 0 \,. \tag{3.7}$$

When passing from δA to $\delta \hat{A}$, which is needed in order to get the leading term,

[5] The unconventional $-$ sign is due to the fact that we chose an anti-hermitean basis for the $u(n)$ generators.

[6] For our purposes it is sufficient to study the spectrum of δA_α as the spectrum of $\delta A_{\bar\alpha}$ does not yield any additional information.

$\Box_{NC}\delta A_\alpha$ correctly, we introduced terms at order α'^4 which are not relevant to this paper. In the previous section we found that the Yang-Mills spectrum was linear in the fieldstrengths f_γ. String theory requires the same spectrum but with the fieldstrength f_γ replaced by $\operatorname{arctanh}(f_\gamma) = f_\gamma + \frac{1}{3}f_\gamma^3 + \frac{1}{5}f_\gamma^5 + \cdots$. It is clear from eq. (3.5) that the effective action provides the correct contribution to the spectrum. In fact it was shown in [22] that knowing the abelian limit and fitting the effective action through this order such that it reproduces the correct spectrum, completely fixes the effective action through this order.

4. Testing the effective action at $\mathcal{O}(\alpha'^3)$

Now we finally turn to the $\mathcal{O}(\alpha'^3)$ correction. In [12] it was found to be $\mathcal{L}_{(3)} = \mathcal{L}_{(3)}^{ND} + \mathcal{L}_{(3)}^{D}$, with,

$$\mathcal{L}_{(3)}^{ND} = -\lambda\,\mathrm{tr}\Big(F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_3}{}^{\mu_4} F_{\mu_5}{}^{\mu_1} F_{\mu_4}{}^{\mu_5} + F_{\mu_1}{}^{\mu_2} F_{\mu_4}{}^{\mu_5} F_{\mu_2}{}^{\mu_3} F_{\mu_5}{}^{\mu_1} F_{\mu_3}{}^{\mu_4} - \\ - \frac{1}{2} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_4}{}^{\mu_5} F_{\mu_3}{}^{\mu_1} F_{\mu_5}{}^{\mu_4} \Big), \tag{4.1}$$

and

$$\mathcal{L}_{(3)}^{D} = \lambda\,\mathrm{tr}\Big(\frac{1}{2} \left(D^{\mu_1} F_{\mu_2}{}^{\mu_3}\right) \left(D_{\mu_1} F_{\mu_3}{}^{\mu_4}\right) F_{\mu_5}{}^{\mu_2} F_{\mu_4}{}^{\mu_5} + \\ + \frac{1}{2} \left(D^{\mu_1} F_{\mu_2}{}^{\mu_3}\right) F_{\mu_5}{}^{\mu_2} \left(D_{\mu_1} F_{\mu_3}{}^{\mu_4}\right) F_{\mu_4}{}^{\mu_5} - \\ - \frac{1}{8} \left(D^{\mu_1} F_{\mu_2}{}^{\mu_3}\right) F_{\mu_4}{}^{\mu_5} \left(D_{\mu_1} F_{\mu_3}{}^{\mu_2}\right) F_{\mu_5}{}^{\mu_4} + \\ + \left(D_{\mu_5} F_{\mu_1}{}^{\mu_2}\right) F_{\mu_3}{}^{\mu_4} \left(D^{\mu_1} F_{\mu_2}{}^{\mu_3}\right) F_{\mu_4}{}^{\mu_5} - \\ - F_{\mu_1}{}^{\mu_2} \left(D^{\mu_1} F_{\mu_3}{}^{\mu_4}\right) \left(D_{\mu_5} F_{\mu_2}{}^{\mu_3}\right) F_{\mu_4}{}^{\mu_5} \Big), \tag{4.2}$$

with $\lambda \in \mathbb{R}$. Matching the derivative terms to those obtained in a direct calculation, [16], we get

$$\lambda = -\frac{2\zeta(3)}{\pi^3}\,. \tag{4.3}$$

In [12], a detailed comparison with other results in the literature, [16], [17] and [18], was made. When converting to the basis we choose for expressing the action, we found that the terms with derivatives all agreed. However at the level of the terms without derivatives no agreement exists between [12], [17] and [18]. In order to check these various results, we will keep the terms with derivatives as in eq. (4.2), but we will replace eq. (4.1) by the most general term without derivatives and leave the coefficients free,

$$
\begin{aligned}
\mathcal{L}_{(3)}^{ND} = \operatorname{tr}\Big(& l_{0,0} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_3}{}^{\mu_4} F_{\mu_4}{}^{\mu_5} F_{\mu_5}{}^{\mu_1} + l_{0,1} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_3}{}^{\mu_4} F_{\mu_5}{}^{\mu_1} F_{\mu_4}{}^{\mu_5} + \\
& + l_{0,2} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_5}{}^{\mu_1} F_{\mu_3}{}^{\mu_4} F_{\mu_4}{}^{\mu_5} + l_{0,3} F_{\mu_1}{}^{\mu_2} F_{\mu_4}{}^{\mu_5} F_{\mu_2}{}^{\mu_3} F_{\mu_5}{}^{\mu_1} F_{\mu_3}{}^{\mu_4} + \\
& + l_{1,0} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_3}{}^{\mu_1} F_{\mu_4}{}^{\mu_5} F_{\mu_5}{}^{\mu_4} + l_{1,1} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_4}{}^{\mu_5} F_{\mu_3}{}^{\mu_1} F_{\mu_5}{}^{\mu_4} \Big), \qquad (4.4)
\end{aligned}
$$

with $l_{0,s},\ l_{1,s} \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Our result corresponds to putting

$$
\begin{aligned}
& l_{0,0} = l_{0,2} = l_{1,0} = 0\,, \\
& l_{0,1} = l_{0,3} = -\lambda\,, \qquad l_{1,1} = \frac{\lambda}{2}\,. \qquad (4.5)
\end{aligned}
$$

Again, we need to expand the effective action through second order in the fluctuations. A straightforward but very tedious calculation yields,

$$
\begin{aligned}
0 = & \left(\Box_{NC} + 2\Big(1 + \frac{1}{3} f_\beta^2\Big) \mathcal{D}_\beta \mathcal{D}_{\bar\beta} - 2i\Big(f_\beta + \frac{1}{3} f_\beta^3\Big) + 4i\Big(f_\alpha + \frac{1}{3} f_\alpha^3\Big) \right) \delta \hat{A}_\alpha + \\
& + \Big(4(4\lambda - x_B) f_\alpha^4 + 2(x_B - 4\lambda) f_\alpha^2 f_\beta^2 + 2(x_B - 4\lambda) f_\alpha f_\beta^3 + \\
& \quad + 2(12\lambda - x_A) f_\beta^4 - 4(\lambda + x_C) f_\beta^2 f_\gamma^2 \Big) \delta A_\alpha + \\
& + 2i(x_B - 4\lambda) f_\beta^2 f_\alpha \mathcal{D}_\beta \mathcal{D}_{\bar\beta} \delta A_\alpha - \\
& - \Big(1 + \frac{1}{3} f_\beta^2 + i(6\lambda - x_A) f_\alpha^3 + i(4\lambda - x_B) f_\alpha^2 f_\beta - 2i(\lambda + x_C) f_\alpha f_\gamma^2 + \\
& \quad + i(x_B - 4\lambda) f_\alpha f_\beta^2 + i(x_A - 12\lambda) f_\beta^3 + 2i(x_C + \lambda) f_\gamma^2 f_\beta \Big) \mathcal{D}_\alpha \mathcal{D}_\beta \delta A_{\bar\beta} - \\
& - \Big(1 + \frac{1}{3} f_\beta^2 + i(6\lambda - x_A) f_\alpha^3 - i(4\lambda - x_B) f_\alpha^2 f_\beta - 2i(\lambda + x_C) f_\alpha f_\gamma^2 + \\
& \quad + i(x_B - 4\lambda) f_\alpha f_\beta^2 - i(x_A - 12\lambda) f_\beta^3 - 2i(x_C + \lambda) f_\gamma^2 f_\beta \Big) \mathcal{D}_\alpha \mathcal{D}_{\bar\beta} \delta A_\beta + \\
& + \lambda f_\gamma^2 \left(\mathcal{D}_\gamma \mathcal{D}_{\bar\gamma} \mathcal{D}_\alpha + \mathcal{D}_{\bar\gamma} \mathcal{D}_\gamma \mathcal{D}_\alpha + \mathcal{D}_\gamma \mathcal{D}_\alpha \mathcal{D}_{\bar\gamma} + \mathcal{D}_{\bar\gamma} \mathcal{D}_\alpha \mathcal{D}_\gamma \right) \left(\mathcal{D}_\beta \delta A_{\bar\beta} + \mathcal{D}_{\bar\beta} \delta A_\beta \right) + \\
& + \mathcal{O}(\alpha'^4)\,, \qquad (4.6)
\end{aligned}
$$

where

$$
\begin{aligned}
\delta \hat{A}_\alpha = & \ \delta A_\alpha + \frac{1}{3} f_\alpha^2 \delta A_\alpha - \frac{1}{6} f_\beta^2 \delta A_\alpha - i(x_A - 4\lambda) f_\alpha^3 \delta A_\alpha - \\
& - 2i(x_C + \lambda) f_\alpha f_\beta^2 \delta A_\alpha + 4i\lambda f_\beta^3 \delta A_\alpha - (\lambda f_\gamma^2 - 2\lambda f_\alpha f_\beta) \mathcal{D}_\alpha \mathcal{D}_\beta \delta A_{\bar\beta} - \\
& - (\lambda f_\gamma^2 + 2\lambda f_\alpha f_\beta) \mathcal{D}_\alpha - \mathcal{D}_{\bar\beta} \delta A_\beta 4\lambda f_\beta^2 - \mathcal{D}_\beta \mathcal{D}_{\bar\beta} \delta A_\alpha + \lambda f_\beta^2 - \mathcal{D}^2 \delta A_\alpha. \qquad (4.7)
\end{aligned}
$$

In order to not complicate our notation unnecessarily, we understand that in the two equations above the indices β and γ are summed over while the index α is kept fixed.

The constants x_A, x_B and x_C are expressed in terms of the coupling constants in eq. (4.4),

$$\begin{aligned} x_A &= 10 l_{0,0} - 2 l_{0,1} + 6 l_{0,2} - 10 l_{0,3} \,, \\ x_B &= -10 l_{0,0} + 6 l_{0,1} + 2 l_{0,2} - 10 l_{0,3} \,, \\ x_C &= 6 l_{1,0} - 2 l_{1,1} \,. \end{aligned} \tag{4.8}$$

It is clear from this result that terms with and without derivatives communicate with each other in a non-trivial way. We now study eq. (4.6) in detail. The first line reproduces already the correct spectrum, so the remainder should vanish. The second and third line of eq. (4.6) would alter the zero-point energy. We get that this term vanishes iff.

$$x_A = 12\lambda \,, \qquad x_B = 4\lambda \,, \qquad x_C = -\lambda \,. \tag{4.9}$$

The fourth line, which would alter the oscillator energy, vanishes then as well. Implementing eq. (4.9) in the remainder of the expression, one finds that it vanishes by virtue of the gauge choice eq. (3.7). So the $\mathcal{O}(\alpha'^3)$ corrections to the effective action, eqs. (4.4) and (4.2) do not alter the spectrum provided eq. (4.9) holds, or equivalently,

$$\begin{aligned} l_{0,0} &= 2\lambda - l_{0,2} + 2 l_{0,3} \,, \\ l_{0,1} &= 4\lambda - 2 l_{0,2} + 5 l_{0,3} \,, \\ l_{1,1} &= \frac{\lambda}{2} + 3 l_{1,0} \,. \end{aligned} \tag{4.10}$$

Using eq. (4.5), one easily checks that the effective action as proposed in [12] indeed reproduces the correct spectrum!

5. Conclusions

We can be quite confident that

$$\begin{aligned} \mathcal{L} = -\frac{1}{4g^2}\,\mathrm{tr}\Bigg(& F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_1} - \frac{1}{24} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_3}{}^{\mu_4} F_{\mu_4}{}^{\mu_1} - \frac{1}{12} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_4}{}^{\mu_1} F_{\mu_3}{}^{\mu_4} + \\ & + \frac{1}{48} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_1} F_{\mu_3}{}^{\mu_4} F_{\mu_4}{}^{\mu_3} + \frac{1}{96} F_{\mu_1}{}^{\mu_2} F_{\mu_3}{}^{\mu_4} F_{\mu_2}{}^{\mu_1} F_{\mu_4}{}^{\mu_3} + \\ & + \frac{2\zeta(3)}{\pi^2} \Bigg(F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_3}{}^{\mu_4} F_{\mu_5}{}^{\mu_1} F_{\mu_4}{}^{\mu_5} + F_{\mu_1}{}^{\mu_2} F_{\mu_4}{}^{\mu_5} F_{\mu_2}{}^{\mu_3} F_{\mu_5}{}^{\mu_1} F_{\mu_3}{}^{\mu_4} - \\ & - \frac{1}{2} F_{\mu_1}{}^{\mu_2} F_{\mu_2}{}^{\mu_3} F_{\mu_4}{}^{\mu_5} F_{\mu_3}{}^{\mu_1} F_{\mu_5}{}^{\mu_4} + F_{\mu_1}{}^{\mu_2} \left(D^{\mu_1} F_{\mu_3}{}^{\mu_4}\right) \left(D_{\mu_5} F_{\mu_2}{}^{\mu_3}\right) F_{\mu_4}{}^{\mu_5} - \\ & - \frac{1}{2} \left(D^{\mu_1} F_{\mu_2}{}^{\mu_3}\right) \left(D_{\mu_1} F_{\mu_3}{}^{\mu_4}\right) F_{\mu_5}{}^{\mu_2} F_{\mu_4}{}^{\mu_5} - \frac{1}{2} \left(D^{\mu_1} F_{\mu_2}{}^{\mu_3}\right) F_{\mu_5}{}^{\mu_2} \left(D_{\mu_1} F_{\mu_3}{}^{\mu_4}\right) F_{\mu_4}{}^{\mu_5} + \\ & + \frac{1}{8} \left(D^{\mu_1} F_{\mu_2}{}^{\mu_3}\right) F_{\mu_4}{}^{\mu_5} \left(D_{\mu_1} F_{\mu_3}{}^{\mu_2}\right) F_{\mu_5}{}^{\mu_4} - \left(D_{\mu_5} F_{\mu_1}{}^{\mu_2}\right) F_{\mu_3}{}^{\mu_4} \left(D^{\mu_1} F_{\mu_2}{}^{\mu_3}\right) F_{\mu_4}{}^{\mu_5} \Bigg)\Bigg), \end{aligned} \tag{5.1}$$

is the full non-abelian open superstring effective action through $\mathcal{O}(\alpha'^3)$. Indeed it is almost uniquely defined by demanding that certain BPS configurations which are known to exist both in the weak field limit as well as in the abelian limit, solve the equations of motion. The only redundancy left was an arbitrary coupling constant at $\mathcal{O}(\alpha'^3)$ which we fixed by comparing the derivative terms in eq. (5.1) to the string theoretic calculation of these terms in [16]. The effective action passes an essential test. Calculating the spectrum in the presence of constant magnetic background fields, correctly reproduces the string theoretic result. We verified that various readings of the direct calculation in [17] does not pass this test.

While the result in eq. (5.1) is not sufficient to make all order predictions, it still assumes the form suggested in [8] as discussed in the introduction to this paper. The proposal of [8] does fix $\mathcal{S}_1$. However, as we showed in this paper, even simple checks of the effective action require not only the knowledge of $\mathcal{S}_1$ but that of $\mathcal{S}_2$ and $\mathcal{S}_3$ as well (see also [11]). While it is highly unlikely to get a closed expression for $\mathcal{S}_2$, one might hope that it is possible for $\mathcal{S}_3$. In order to get more insight in this, we are constructing the next order in the effective action using the method of [13] and [12]. The reader might wonder whether the method of [13] and [12] at high orders is any less involved than a direct superstring scattering amplitude calculation. In fact, both become very complicated. But contrary to a direct calculation, the method of [13] and [12] lends itself for a computerized implementation.

Finally let us turn to the calculation in [18]. There the terms of the same dimensions as the ones discussed in this paper (F^5 and D^2F^4), in the one-loop effective action of $N = 4$ supersymmetric Yang-Mills in four dimensions were calculated. Using the conversion table in [12], makes it possible to pass from their basis for the action to ours. The reader can verify for himself that when doing this the resulting structure does not satisfy eq. (4.10). This is a manifestation of the fact that in general one should not expect a direct relation between the tree-level open string effective action and the quantum super Yang-Mills effective action (for a more detailed discussion, we refer to [23]). In particular, already in the abelian case, it is not too hard to see that the F^8 term in the one-loop $N = 4$ super Yang-Mills effective action is different in structure, [23], from the F^8 term in the Born-infeld action [24].

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